SDOF & MDOF Dynamics Cheat Sheet

VS&A All Day | Vibroacoustics Engineering Reference

SDOF Fundamentals

Equation of Motion

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Standard Form: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t)/m$

Key Parameters

Parameter	Symbol	Formula	Units
Natural frequency	ω_n	$\sqrt{(k/m)}$	rad/s
Natural frequency	f_n	$\omega_n/(2\pi) = (1/2\pi)\sqrt{(k/m)}$	Hz
Period	T	$1/f_n = 2\pi/\omega_n$	s
Damping ratio	ζ	$c/(2m\omega_n) = c/(2\sqrt{km})$	—
Critical damping	c_cr	$2m\omega_n = 2\sqrt{(km)}$	N·s/m
Quality factor	Q	$1/(2\zeta) = m\omega_n/c$	—
Damped frequency	ωd	$\omega_n\sqrt{(1-\zeta^2)}$	rad/s
Log decrement	δ	$\ln(x_n/x_{n+1}) = 2\pi\zeta/\sqrt{(1-\zeta^2)}$	—

Damping Regimes

Condition	ζ Value	Behavior
Underdamped	$\zeta < 1$	Oscillates with decay
Critically damped	$\zeta = 1$	Fastest return, no overshoot
Overdamped	$\zeta > 1$	Slow return, no oscillation

Free Vibration Response

Underdamped ($\zeta < 1$)

$$x(t) = Xe^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

Critically Damped ($\zeta = 1$)

$$x(t) = (A + Bt)e^{-\omega_n t}$$

Overdamped ($\zeta > 1$)

$$x(t) = Ae^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} + Be^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t}$$

Forced Vibration Response

Frequency Ratio

$$r = \omega/\omega_n = f/f_n$$

Dynamic Magnification Factor

$$DMF = \frac{X}{X_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Phase Angle

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

Key Values

Condition	r	DMF	Phase
Static	0	1	0°
Resonance	1	Q = 1/(2ζ)	90°
High frequency	>>1	≈ $1/r^2$	180°

Transmissibility

Force Transmissibility (Fixed Base)

$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Displacement Transmissibility (Base Excitation)

$$TR = \frac{X}{Y} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Isolation Region

- **TR < 1** when **r > $\sqrt{2}$** ($f > 1.41 f_n$)
- Lower damping = better isolation at high r
- Higher damping = lower resonance peak

Random Vibration

Miles' Equation

$$G_{rms} = \sqrt{\frac{\pi}{2} f_n Q \cdot W_0}$$

or equivalently: $G_{rms} = \sqrt{\frac{\pi}{4\zeta} f_n \cdot W_0}$

Where:

- W_0 = input PSD (g^2/Hz), assumed white noise
- f_n = natural frequency (Hz)
- Q = quality factor = 1/(2ζ)

Key Assumptions

1. White noise (flat) input PSD
2. Light damping ($\zeta \ll 1$)
3. Linear system
4. Stationary random process

3σ Design

- 99.7% of peaks within $\pm 3\sigma$
- Peak = $3 \times$ Grms

Damping Types

Type	Force Law	Energy/Cycle	Decay
Viscous	$F = c\dot{x}$	$\pi c \omega X^2$	Exponential
Structural	$F = \eta k \lvert x \rvert \mathrm{sgn}(\dot{x})$	$\pi \eta k X^2$	Exponential
Coulomb	$F = \mu N \cdot \mathrm{sgn}(\dot{x})$	$4\mu N X$	Linear

Equivalent Viscous Damping

$$c_{eq} = \frac{\eta k}{\omega}$$

Complex Stiffness

$$k^* = k(1 + i\eta)$$

Loss Factor to Damping Ratio

$$\eta = 2\zeta$$

(at resonance)

Damping Measurement

Half-Power Bandwidth

$$\zeta = \frac{\Delta f}{2f_n} = \frac{f_2 - f_1}{2f_n}$$

Where f_1, f_2 are frequencies at -3dB (1/√2 of peak)

Log Decrement

$$\delta = \ln\left(\frac{x_n}{x_{n+1}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

For light damping: $\delta \approx 2\pi\zeta$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \approx \frac{\delta}{2\pi}$$

MDOF Systems

Matrix Equation of Motion

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$

Eigenvalue Problem

$$[K]\{\phi\} = \omega^2[M]\{\phi\}$$

$$\det([K] - \omega^2[M]) = 0$$

Orthogonality

$$\{\phi\}_i^T[M]\{\phi\}_j = 0 \quad (i \neq j)$$

$$\{\phi\}_i^T[K]\{\phi\}_j = 0 \quad (i \neq j)$$

Modal Parameters

$$M_i = \{\phi\}_i^T[M]\{\phi\}_i$$

(generalized mass) \$$K_i = \{\phi\}_i^T[K]\{\phi\}_i (generalized stiffness) \omega_i^2 = K_i/M_i$\$

Modal Superposition

$$\{x(t)\} = \sum_{i=1}^{N}\{\phi\}_i q_i(t)$$

Decoupled equation for mode i: \$$M_i\ddot{q}_i + C_i\dot{q}_i + K_i q_i = \{\phi\}_i^T\{F(t)\}$\$

Modal Participation Factor

$$\Gamma_i = \frac{\{\phi\}_i^T[M]\{r\}}{M_i}$$

Effective Modal Mass

$$M_{eff,i} = \frac{(\{\phi\}_i^T[M]\{r\})^2}{M_i}$$

$$\sum_{i=1}^{N} M_{eff,i} = \text{Total Mass}$$

Rayleigh Damping

$$[C] = \alpha[M] + \beta[K]$$

Modal Damping Ratio

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

Coefficients (given ζ at ω_1 and ω_2)

$$\alpha = \frac{2\omega_1\omega_2(\zeta_1\omega_2 - \zeta_2\omega_1)}{\omega_2^2 - \omega_1^2}$$

$$\beta = \frac{2(\zeta_2\omega_2 - \zeta_1\omega_1)}{\omega_2^2 - \omega_1^2}$$

If $\zeta_1 = \zeta_2 = \zeta$: $\alpha = \frac{2\zeta\omega_1\omega_2}{\omega_1+\omega_2}, \quad \beta = \frac{2\zeta}{\omega_1+\omega_2}$

Unit Conversions

From	To	Multiply by
rad/s	Hz	1/(2π) ≈ 0.159
Hz	rad/s	2π ≈ 6.283
g	m/s^2	9.81
m/s^2	g	0.102
in	m	0.0254
lb	N	4.448
lbf/in	N/m	175.1

Common Values

Typical Damping Ratios

Material/System	ζ (%)
Steel structures	1-2
Aluminum structures	0.5-1
Bolted joints	2-5
Welded joints	1-3
Rubber mounts	5-15
Fluid dampers	10-30

Typical Q Factors

Q	ζ	Description
100	0.5%	Very lightly damped
50	1%	Lightly damped
20	2.5%	Moderately damped
10	5%	Well damped
5	10%	Heavily damped